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Comment on the Run Time Statistics in Models of Growth in Disordered Media

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We point out an error in the equations for the evolution of the run time statistics given in a paper by Marsili and give the correct expression. Moreover, we discuss the *annealing* approximation which is implicit in the transformation.

KEY WORDS: Irreversible growth processes; quenched disorder; self-organized criticality; invasion percolation; conditional probability.

In a recent paper,[1] hereafter referred to as paper I, Marsili proposed a transformation of a quenched process into an annealed one, along lines originally proposed in ref. 2. The class of models considered are based on a dynamics which is determined by the extreme of a quenched random field ε_i, where i labels a site in a d dimensional lattice. These models include invasion percolation as well as many other models recently introduced[3]. The purpose of this comment is to correct an error in the equations for the evolution of the *run time statistics* (hereafter RTS), which is the effective probability distribution of disorder variables at a time t, and to clarify the nature of the approximation involved. Briefly (for a more complete account of the transformation see paper I), the process evolves according to the following rule: at the generic time t, the dynamics has produced a cluster of occupied sites $\mathscr{C}_t$. The next site to be occupied lies on a properly defined perimeter $\partial\mathscr{C}_t$ of $\mathscr{C}_t$. This site is chosen to be the one with the smallest value of the disorder variable ε_i for $i \in \partial\mathscr{C}_t$. Since the disorder is assigned initially to each site i, once the initial cluster $\mathscr{C}_0$ is specified, the process is deterministic, The transformation of this process to a stochastic one, which has

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clear advantages which are discussed in I, goes as follows: Let $\tau_{i,t}$ be the time elapsed since site i first entered the perimeter set: $i \in \partial\mathscr{C}_{t'}$ with $t' = t - \tau_{i,t}, \ldots, t$ and $i \notin \partial\mathscr{C}_{t'}$ for $t' < t - \tau_{i,t}$. Imagine that the effective distribution of the variables ε_i on sites in $\partial\mathscr{C}_t$ is given. This distribution, which depends on $\tau_{i,t}$, is the RTS: $p_{\tau_{i,t},t}(x)\,dx \equiv P\{x \leqslant \varepsilon_i < x + dx\}$. This allows us to compute the probability that site i will be selected [see Eq. (1) in I]:

$$\mu_{\tau,t} = \int_0^1 p_{\tau,t}(x) \prod_k [1 - P_{k,t}(x)]^{n_{k,t} - \delta_{k,\delta}} \tag{1}$$

where $n_{\tau,t}$ is the number of sites with $\tau_{i,t} = \tau$ and $P_{k,t}(x) = \int_0^x p_{k,t}(y)\,dy$. Indeed Eq. (1) yields the probability of the event $\{\varepsilon_i \leqslant \varepsilon_j \; \forall j \in \partial\mathscr{C}_t\}$ in which ε_i is the smallest variable in $\partial\mathscr{C}_t$. With probability $\tilde{\mu}$, t, site i with $\tau_{i,t} = \tilde{\tau}$ will be the smallest. This implies that its probability distribution is not $p_{\tilde{\tau},t}(x)$, but rather

$$m_t(x \mid \tilde{\tau}) = \frac{1}{\mu_{\tilde{\tau},t}} p_{\tilde{\tau},t}(x) \prod_k [1 - P_{k,t}(x)]^{n_{k,t} - \delta_{k,\tilde{\tau}}} \tag{2}$$

[Eq. (3) in I], which accounts, conditionally, for the fact that $\varepsilon_i \leqslant \varepsilon_j$ for all $j \in \partial\mathscr{C}_t$. Again the rules of conditional probability can be used to find the RTS $p_{k+1,t+1}(x)$ at the following time step:

$$p_{k+1,t+1}(x) = p_{k,t}(x) \int_0^x \frac{m_t(y \mid \tilde{\tau})}{1 - P_{k,t}(y)}\,dy \tag{3}$$

Note that this differs from the analogous equation (4) in paper I, which should be replaced by the above one. Before sketching the derivation of Eq. (3) we note that (i) $P_{k+1,t+1}(x)$ is properly normalized, as can be explicitly checked by using Eqs. (1) and (2); (ii) $p_{k+1,t+1}(x)$ is linear in $p_{k,t}(x)$, whereas in eq. (4) of paper I $p_{k,t}(x)$ appeared twice in the equation for $p_{k+1,t+1}(x)$ [explicitly in front of $M_t(x \mid \tilde{\tau})$ and again in $M_t(x \mid \tilde{\tau})$ itself, the denominator in the integral of Eq. (3) explicitly cancels this second occurrence] (iii) the substitution of Eq. (4) in paper I with Eq. (3) does not affect the results derived in Sections 3–5 of paper I, even though it modifies slightly the numerical results found in direct calculation based on the RTS and real-space methods for critical phenomena, as in refs. 4 and 5.

In order to derive Eq. (3), let us call $A_j(t; x)$ the event $\{x \leqslant \varepsilon_j < x + dx$ at time $t\}$ and $B_i(t)$ the event $\{\varepsilon_i < \varepsilon_j \; \forall j \in \partial\mathscr{C}_t\}$. Therefore

$$p_{\tau_j,t+1,t+1}(x)\,dx = P\{A_j(t; x) \mid B_i(t)\} \tag{4}$$

where $\tau_{j,t+1} = \tau_{j,t} + 1$. Here $P\{A|B\}$ denoted the probability of event A conditional to the occurrence of B, and is given by $P\{A|B\} = P\{AB\}/P\{B\}$. In addition, AB is the event in which both A and B occur, and in our case it is easy to find that

$$P\{A_j(t;x)\, B_i(t)\}$$

$$= dx\, p_{k_j,t}(x) \int_0^x dy\, p_{\tilde{\tau},t}(y) \prod_k [1 - P_{k,t}(y)]^{n_{k,t} - \delta_{k,\tilde{\tau}} - \delta_{k,\tau_j,t}} \qquad (5)$$

Indeed the first factor is just the probability of $A_j(t;x)$. The integral runs only up to x because $y = \varepsilon_i < \varepsilon_j = x$ at time t (ε_i is the smallest variable in $\partial\mathscr{C}_t$). The product inside the integral expresses that also all the other variables in $\partial\mathscr{C}_t$ must be smaller than $\varepsilon_i = y$. Note that the site i (for which $\tau_{i,t} = \tilde{\tau}$) and the site j must be excluded from this product, which is the reason for the exponent $n_{k,t} - \delta_{k,\tilde{\tau}} - \delta_{k,\tau_j,t}$. Therefore this is the correct formula that expresses the probability of the *simultaneous* occurrence of the events $A_j(t;x)$ and $B_i(t)$. Finally we note that $P\{B_i(t)\} = \mu_{\tilde{\tau},t}$ as given in Eq. (1). Combining this with Equations (4) and (5) and using Eq. (2), one easily arrives at Eq. (3).

We note, moreover, that the transformation from the quenched dynamics to the annealed one is not exact. The approximation lies in the assumption of independence between the variables in $\partial\mathscr{C}_t$. The event of selection of the smallest random variables modifies the statistics of the remaining variables as a whole. In other words, one can define an exact transformation only by dealing with the full joint probability distribution $\rho_t: (x_1, ..., x_{N_t})$ of the variables in $\partial\mathscr{C}_t$. It can be easily checked that, even if $\rho_t(x_1, ..., x_{N_t}) = \prod_i \rho_{i,t}(x_t)$ at time t, after one step of evolution the joint probability distribution is no longer a product over single-variable distributions. In Eq. (3) the other Variables in $\partial\mathscr{C}_t$ are integrated over, which means that this equation holds for the marginal distributions of one variable. In other words, the condition $\varepsilon_j < \min(\varepsilon_i)$ is applied independently to each variable to update its distribution.

Note that the variables ε_i were initially chosen as independent. It is, however, well known that if one orders N variables ε_i in ascending order $\varepsilon_{p_i} < \varepsilon_{p_{i+1}}$, where p_i is a permutation of the indices 1,..., N, the variables ε_{p_i} are no longer independent. In extreme dynamics the correlation arises for exactly the same reason. There the order relations are established between variables which are selected and variables which remain on the perimeter set.

In order to illustrate this point and to find a rough estimate of the correlation, take n independent uniform variables ε_i and impose on them the condition $\varepsilon_i \geqslant \varepsilon_0$ for all $i = 1,..., n$ [or equivalently $\min_i(\varepsilon_i) > \varepsilon_0$]. Then the distribution of the ε_i is no longer that of independent variables. The (marginal) distribution of one of the variables is

$$p_1(x) = \frac{n+1}{n}[1-(1-x)^n]$$

while that of a pair of variables is

$$p_2(x, y) = \frac{n+1}{n-1}\{\theta(y-x)[1-(1-x)^{n-1}] + \theta(x-y)[1-(1-y)^{n-1}]\}$$

With these we can calculate the averages $\langle \varepsilon_i \rangle$ and $\langle \varepsilon_i \varepsilon_j \rangle$ for $i \neq j$, and therefore evaluate the correlation

$$\langle \varepsilon_i \varepsilon_j \rangle - \langle \varepsilon_i \rangle \langle \varepsilon_j \rangle = \frac{n+1}{4(n+2)^2(n+3)} \sim \frac{1}{n^2} \tag{6}$$

This suggests that if one writes

$$\rho_t(x_1,..., x_n) = \prod_{i=1}^{n} \rho_{i,t}(x_i) + \prod_{i \neq j} \rho^{(2)}_{ij,t}(x_i, x_j) + \cdots$$

then the second term is negligible for large n. In other words the first term in the above equation produces a contribution of order $1/n^2$ to the second term at each step. Taking $n = N_t$ suggests that the correlation between variables is negligible in the asymptotic limit $N_t \to \infty$. A more careful study is necessary, however, to settle the question in a definite way.

The interpretation of the stochastic process resulting from the application of the RTS is therefore the following: at each time step, every variable ε_i is extracted from its distribution $\rho_{i,t}(x)$, then the minimum variable is found, and finally the distribution of the variables is updated. In the quenched process, instead, the variables are fixed from the beginning. The update rule of the RTS is, however, such that the statistical properties of the variable which is extracted at time t are the same as those of a variable which would have witnessed the actual quenched process. This *annealing approximation* to the quenched process includes only time correlations between variables, whereas space correlations, i.e., correlations between variables ε_i at the same time, are neglected.

REFERENCES

1. M. Marsili, *J. Stat. Phys.* **77**:733 (1994).
2. L. Pietronero and W. R. Schneider, *Physica A* **170**:81 (1990).
3. R. Lenormand and S. Bories, *Computer Rendus. Acad. Sci.* **291**:279 (1980); K. Sneppen, *Phys. Rev. Lett.* **69**:3539 (1992); **71**:101 (1993); P. Bak and K. Sneppen, *Phys. Rev. Lett.* **71**:4083 (1993).
4. M. Marsili, *Europhys. Lett.* **28**:385 (1994).
5. A. Gabrielli, M. Marsili, R. Cafiero, and L. Pietronero, to be published.